Fourier Analysis 04-09.

Review.

Thm (Fourier Inversion formula)

Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

Thm 1 (Plancherel formula).

Let $f \in M(\mathbb{R})$. Suppose $\hat{f} \in M(\mathbb{R})$.

Then we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$$

Proof. Let $h(x) = \overline{f(-x)}$. Then $h \in M(\mathbb{R})$.

Then

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} \overline{f(-x)}\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} \overline{f(-x) \cdot e^{2\pi i \xi x}}\, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(-x) e^{2\pi i \xi x} dx}$$

$$\underset{\text{Letting } y=-x}{=} \overline{\int_{\infty}^{-\infty} f(y)\, e^{-2\pi i \xi y} (-)\, dy}$$

$$= \overline{\hat{f}(\xi)}.$$

Let us consider $f * h$.

Notice that $f * \tilde{h} \in \mathcal{M}(\mathbb{R})$.

$$\widehat{f * \tilde{h}}(\xi) = \hat{f}(\xi) \cdot \hat{\tilde{h}}(\xi) = \hat{f}(\xi) \cdot \overline{\hat{f}(\xi)}$$

$$= |\hat{f}(\xi)|^2$$

Hence $\widehat{f * \tilde{h}} \in \mathcal{M}(\mathbb{R})$.

Now applying Fourier Inversion formula to $f * \tilde{h}$, we obtain

$$f * \tilde{h}(0) = \int_{-\infty}^{\infty} \widehat{f * \tilde{h}}(\xi)\, d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi.$$

Notice that

$$f * \tilde{h}(0) = \int_{-\infty}^{\infty} f(x)\, \tilde{h}(-x)\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \overline{f(x)}\, dx$$

$$= \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

So we obtain

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi.$$

$\boxed{}$

Schwartz space:

Let $S(\mathbb{R})$ denote the collection of all C^∞ functions f on $\mathbb{R}$ such that for any $k, \ell \geq 0$,

$$\sup_{x \in \mathbb{R}} |x|^k \cdot |f^{(\ell)}(x)| < \infty.$$

$$\left(\quad |f^{(\ell)}(x)| \leq \frac{C}{1+|x|^k} \right)$$

- $S(\mathbb{R})$ is a vector space over $\mathbb{C}$
- $\forall f \in S(\mathbb{R})$, $f' \in S(\mathbb{R})$.

Prop 2: $f \in S(\mathbb{R}) \Leftrightarrow \hat{f} \in S(\mathbb{R})$.

Pf. We only prove the direction:

$$f \in S(\mathbb{R}) \Rightarrow \hat{f} \in S(\mathbb{R}).$$

Suppose $f \in S(\mathbb{R})$.

We need to show that

$$\sup_{\xi \in \mathbb{R}} |\xi|^k \cdot \left| \hat{f}^{(\ell)}(\xi) \right| < \infty \quad \forall\ k, \ell \geq 0.$$

Notice that

$$(-2\pi i x)^\ell f(x) \xrightarrow{\ \tilde{f}\ } \hat{f}^{(\ell)}(\xi)$$

$$F(x) := \frac{d^k\left(\left(-2\pi i x\right)^{\ell} f(x)\right)}{dx^k} \xrightarrow{\mathcal{F}} \left(2\pi i \xi\right)^k \cdot \hat{f}^{(\ell)}(\xi)$$

In particular

$$\sup_{\xi\in\mathbb{R}} \left| \left(2\pi i \xi\right)^k \hat{f}^{(\ell)}(\xi) \right| \leq \int_{-\infty}^{\infty} \left| \quad F(x) \quad \right| dx$$

$$< \infty \qquad \left(\text{since } F \in \mathcal{S}(\mathbb{R})\right)$$

$$\Rightarrow \qquad \sup_{\xi\in\mathbb{R}} |\xi|^k \cdot \left|\hat{f}^{(\ell)}(\xi)\right| < \infty .$$

$\boxed{/\!/}$

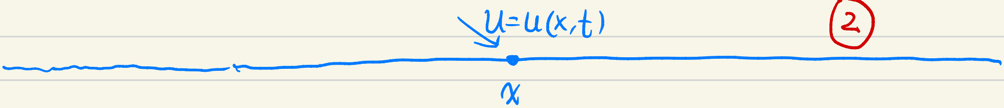
§ 5.5. <u>Application 1</u>: The time-dependent heat equation on the real line.

Consider the heat equation

- $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \quad t > 0. \qquad 1$$

(where $u = u(x,t)$ —— temperature at the location x at time t)

- $$u(x,0) = f(x), \quad x \in \mathbb{R} \quad \text{(initial condition)} \qquad 2$$



We first find a solution by a formal argument:

Taking the Fourier transform on both sides of 1 (with respect to x).

$$\frac{\partial \hat{u}(\xi,t)}{\partial t} = (2\pi i \xi)^2 \, \hat{u}(\xi,t)$$

$$= -4\pi^2 \xi^2 \, \hat{u}(\xi,t).$$

$$\left(\int_{\mathbb{R}} \frac{\partial u(x,t)}{\partial t} e^{-2\pi i \xi x} dx \right.$$

$$= \frac{\partial}{\partial t} \int_{\mathbb{R}} u(x,t) e^{-2\pi i \xi x} dx$$

$$= \frac{\partial}{\partial t} \hat{u}(\xi, t).$$

$$\int_{\mathbb{R}} \frac{\partial^2 u(x,t)}{\partial^2 x} e^{-2\pi i \xi x} dx$$

$$= (2\pi i \xi)^2 \cdot \int_{\mathbb{R}} u(x,t) e^{-2\pi i \xi x} dx$$

$$= -4\pi^2 \xi^2 \hat{u}(\xi, t).$$

Hence we obtain a first order Linear ODE

$$\frac{d}{dt} \hat{u}(\xi,t) = -4\pi^2 \xi^2 \hat{u}(\xi,t).$$

Fix ξ, we see that

$$\hat{u}(\xi,t) = A(\xi)\cdot e^{-4\pi^2\xi^2 t}$$

(taking $t=0$)

Observe that

$$\hat{u}(\xi,0) = \hat{f}(\xi)$$

Hence we have $A(\xi) = \hat{f}(\xi)$.

Now we obtain

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cdot e^{-4\pi^2\xi^2 t}$$

Notice that if setting

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{x^2}{4t}}, \quad t>0,\ x\in\mathbb{R}.$$

Check : $\widehat{\mathcal{H}_t}(\xi) = e^{-4\pi^2\xi^2 t}$.

Recall $e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi\xi^2}$.

$$e^{-\frac{x^2}{4t}} = e^{-\pi\left(\frac{x}{\sqrt{4\pi t}}\right)^2} \xrightarrow{\mathcal{F}} \sqrt{4\pi t}\cdot e^{-\pi\xi^2\cdot(4\pi t)}$$

$$= \sqrt{4\pi t}\, e^{-4\pi^2\xi^2 t}$$

We call $\{\mathcal{H}_t(x)\}$ the heat kernel on the real line.

By the above analysis, we see that

$$\hat{u}(\xi,t) = \widehat{f * \mathcal{H}_t}(\xi).$$

Now by the inversion formula, we see that

$$\boxed{u(x,t) = f * \mathcal{H}_t(x).}$$

Next we conduct a theoretic check.

Thm 3. Let $f \in S(\mathbb{R})$. Let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then

① $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$

② $u(x,t) \rightrightarrows f(x)$ as $t \to 0$

③ $\int_{\mathbb{R}} |u(x,t) - f(x)|^2 dx \to 0$ as $t \to 0$.

Pf. Since $u = f * \mathcal{H}_t$, both $f, \mathcal{H}_t \in S(\mathbb{R})$, it is not hard to show that

$$f * \mathcal{H}_t \in S(\mathbb{R}) \quad \forall\, t > 0.$$

Now by Fourier inversion formula,

$$u(x,t) = \int_{-\infty}^{\infty} \hat{u}(\xi, t)\, e^{2\pi i \xi x} dx$$

(*) $$= \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} e^{2\pi i \xi x} d\xi$$

Let us show

$$\frac{\partial u}{\partial t} \text{ exists}$$

$$\frac{u(x, t+\Delta t) - u(x,t)}{\Delta t} = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot \frac{e^{-4\pi^2\xi^2(t+\Delta t)} - e^{-4\pi^2\xi^2 t}}{\Delta t} \cdot e^{2\pi i \xi x} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \cdot \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} e^{2\pi i \xi x} d\xi$$

Notice that

$$\left| \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} \right| \leq \text{Const} \cdot |\xi^2|$$

$$\lim_{\Delta t \to 0} \frac{e^{-4\pi^2\xi^2\Delta t} - 1}{\Delta t} = (-4\pi^2\xi^2)$$

By Lebesgue's dominated Convergence Thm,

$$\lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2 \xi^2 t} \frac{e^{-4\pi^2\xi^2 \Delta t} - 1}{\Delta t} \cdot e^{2\pi i \xi x} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \cdot (-4\pi^2 \xi^2) \cdot e^{2\pi i \xi x} d\xi$$

By similarly arguments, we see that

$$u \in C^\infty(\mathbb{R} \times \mathbb{R}_+).$$

By (*), we have $\frac{\partial^2 u}{\partial x^2} = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \cdot e^{2\pi i \xi x} (2\pi i \xi)^2 d\xi$

and $\frac{\partial u}{\partial t} = \int_{-\infty}^{\infty} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} (-4\pi^2 \xi^2) e^{2\pi i \xi x} d\xi.$

So $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$. This proves 1.

Part 2 is a consequence that

$$\{\mathcal{H}_t\}_{t>0}$$

is a good kernel on $\mathbb{R}$.

Below we prove ③:

By Plancherel formula,

$$\int_{-\infty}^{\infty} | u(x,t) - f(x) |^2 dx$$

$$= \int_{-\infty}^{\infty} | \hat{u}(\xi, t) - \hat{f}(\xi) |^2 d\xi$$

$$= \int_{-\infty}^{\infty} | \hat{f}(\xi) \cdot e^{-4\pi^2 \xi^2 t} - \hat{f}(\xi) |^2 d\xi$$

$$= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \cdot | e^{-4\pi^2 \xi^2 t} - 1 |^2 d\xi$$

Notice that

$$|\hat{f}(\xi)|^2 \cdot |e^{-4\pi^2 \xi^2 t} - 1|^2 \leq |\hat{f}(\xi)|^2$$

By DCT,

$$\lim_{t \to 0} \int |\hat{f}(\xi)|^2 \left| e^{-4\pi^2 \xi^2 t} - 1 \right|^2 d\xi$$

$$= \int_{\mathbb{R}} \lim_{t \to 0} |\hat{f}(\xi)|^2 \cdot \left| e^{-4\pi^2 \xi^2 t} - 1 \right|^2 d\xi$$

$$= 0.$$

$\blacksquare$